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1994 J. Phys. A: Math. Gen. 27 L559

(http://iopscience.iop.org/0305-4470/27/15/005)

J. Phys. A: Math. Gen. 27 (1994) L559–L562. Printed in the UK

LETTER TO THE EDITOR

Renormalization-group analysis of turbulent transport

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Received 7 June 1994

Abstract. The turbulent transport is investigated by the renormalization-group method. It is shown that the deviation from the Gaussian distribution for the turbulent velocity has important effects upon the advection of a passive scalar. Expanding the equation of turbulent transport up to the higher order, the Batchelor constant is predicted to be Ba = 0.761 in agreement with the experimental data 0.5–0.8.

Yakhot and Orszag [1–4] have applied the renormalization-group (RG) method developed by Forster, Nelson and Stephen [5] to analyse the turbulent transport problem. Expanding the equation of turbulent transport up to the second order, they have calculated the Batchelor constant Ba=1.161. However, the result is not in agreement with the experimental data because the experimental data for the Batchelor constant defined in the RG theory of turbulence is 0.5–0.8 [6].

Though the random force introduced in the RG theory of turbulence obeys the Gaussian distribution, the turbulent velocity deviates from the Gaussian distribution as a result of the nonlinear mode coupling. Yakhot and Orszag [1–4] have analysed the advection of a passive scalar by the turbulent velocity part of the Gaussian distribution. In order to consider the effect of the turbulent velocity part of the non-Gaussian distribution upon the advection of a passive scalar, we expand the equation of turbulent transport up to the higher order utilizing the Navier–Stokes equations. The Batchelor constant is predicted to be Ba=0.761 in agreement with the experimental data 0.5–0.8.

Introducing the Fourier decomposition of the velocity fields with an ultraviolet cut-off $\Lambda = O(k_{\mathrm{d}})$, where k_{d} is the Kolmogrov dissipation wavenumber, the d-dimensional Fourier-transformed Navier–Stokes equations with a random force term for incompressible flow are

$$v_l(\hat{k}) = G^0(\hat{k}) f_l(\hat{k}) - \frac{\mathrm{i}\lambda_0}{2} G^0(\hat{k}) P_{lmn}(k) \int_{q<\Lambda} v_m(\hat{q}) v_n(\hat{k}-\hat{q}) \frac{\mathrm{d}\hat{q}}{(2\pi)^{d+1}} \qquad (1)$$

where the random force is specified by the two-point correlation:

$$\langle f_i(\hat{k}) f_j(\hat{k}_l) \rangle = 2(2\pi)^{d+1} D_0 k^{-y} P_{ij}(k) \delta(\hat{k}+\hat{k}_1). \qquad (2)$$

Here

$$G^0(\hat{k}) = (-\mathrm{i}\omega + \nu_0 k^2)^{-1}$$

$$P_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2}$$

$$P_{lmn}(k) = k_n P_{lm}(k) + k_m P_{ln}(k).$$

$\hat{k} = (k, \omega)$, $\lambda_0(=1)$ is the unrenormalized expansion parameter. Eliminating the modes $\nu^>$ belonging to the wavenumber band $\Lambda e^{-r} < q < \Lambda$ from the equations of the motion for the modes $\nu^<$ belonging to the wavenumber $q < \Lambda e^{-r}$, the effects of the eliminated modes can be taken into account in terms of the renormalized viscosity [7]:

$$\nu(r) = \nu_0 \left[1 + \frac{3}{\varepsilon} A_d \frac{\lambda_0^2 D_0}{\nu_0^3 \Lambda^\varepsilon} (e^{\varepsilon r} - 1) \right]^{1/3} \tag{3}$$

where

$$A_d = \tilde{A}_d \frac{S_d}{(2\pi)^d} \qquad \tilde{A}_d = \frac{d^2 - d}{2d(d+2)} \qquad \varepsilon = 4 + y - d. \tag{4}$$

Based upon the Yakhot–Orszag work, the RG method is applied to the problem of the distribution of a passive scalar advected by a turbulent fluid in the present work. The equation of motion for the Fourier components of a passive scalar $T(\hat{k})$ is

$$T(\hat{k}) = -i\lambda_0' g^0(\hat{k}) k_l \int v_l(\hat{q}) T(\hat{k} - \hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}. \tag{5}$$

Here $g^0(\hat{k}) = (-i\omega + \chi_0 k^2)^{-1}$, $\lambda_0'(=1)$ is the unrenormalized expansion parameter. Utilizing the equations (1) and (5) to remove the modes $T^>$ belonging to the wavenumber band $\Lambda e^{-r} < q < \Lambda$, we write the equation of motion for the modes $T^<$ belonging to the wavenumber band $q < \Lambda e^{-r}$ up to the higher order in λ_0 and λ_0':

$$\begin{aligned}
(-i\omega + \chi_0 k^2) T^<(\hat{k}) &= -i\lambda_0' k_l \int v_l^<(\hat{q}) T^<(\hat{k} - \hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} - 2\lambda_0'^2 D_0 T^<(\hat{k}) k_l k_n \\
&\times \int_{\Lambda e^{-r} < q < \Lambda} |G^0(\hat{q})|^2 g^0(\hat{k} - \hat{q}) P_{ln}(q) q^{-y} \frac{d\hat{q}}{(2\pi)^{d+1}} + 2i\lambda_0'^3 D_0 k_l \\
&\times \int_{\Lambda e^{-r} < q < \Lambda} (k_n - q_n)(k_m - p_m) |G^0(\hat{q})|^2 g^0(\hat{k} - \hat{q}) g^0(\hat{k} - \hat{q} - \hat{p}) P_{lm}(q) q^{-y} v_n^<(\hat{p}) \\
&\times T^<(\hat{k} - \hat{p}) \frac{d\hat{q}\, d\hat{p}}{(2\pi)^{2d+2}} + 2i\lambda_0 \lambda_0'^2 D_0 k_l \int_{\Lambda e^{-r} < q < \Lambda} (k_m - j_m) |G^0(\hat{q})|^2 G^0(\hat{j}) g^0(\hat{k} - \hat{j}) \\
&\times P_{l\beta n}(j) P_{\beta m}(q) q^{-y} v_n^<(\hat{q} + \hat{j}) T^<(\hat{k} - \hat{q} - \hat{j}) \frac{d\hat{q}\, d\hat{j}}{(2\pi)^{2d+2}}.
\end{aligned} \tag{6}$$

In the limit $k \to 0$ and $\omega \to 0$, the second term on the right side of (6) gives the correction to the bare diffusivity [1–3]:

$$\Delta\chi_1(0) = \frac{d-1}{d} \frac{S_d}{(2\pi)^d} \frac{\lambda_0'^2 D_0}{\nu_0(\chi_0 + \nu_0)\Lambda^\varepsilon} \frac{e^{\varepsilon r} - 1}{\varepsilon}. \tag{7}$$

Only considering the effects of the second term on the right side of (6), Yakhot and Orszag [1–4] have calculated the Batchelor constant Ba=1.161. In the present work, the effects of the third and fourth terms on the right side of (6) are analysed. The corrections to the bare diffusivity given by the two terms are evaluated to be

$$\Delta\chi_2(0) = (-2A + B) \frac{2\lambda_0'^3}{d(d+2)} \frac{\chi_0 D_0 \frac{S_d}{(2\pi)^d}}{\nu_0(\nu_0 + \chi_0)^3 \Lambda^{\varepsilon+2}} \frac{e^{(\varepsilon+2)r} - 1}{\varepsilon + 2} \tag{8}$$

and

$$\Delta\chi_3(0) = \{-[F_1(\chi_0, \nu_0) + F_2^{(d)}(\chi_0, \nu_0)]A + F_2^{(d)}(\chi_0, \nu_0)B\} \times \frac{\lambda_0\lambda_0'^2}{2d(d+2)\Lambda^{\varepsilon+2}} D_0 \frac{S_d}{(2\pi)^d} \frac{e^{(\varepsilon+2)r} - 1}{\varepsilon+2} \tag{9}$$

where

$$A = \frac{ik_l k_n}{k^2 T^<(\hat{k})} \int p_l v_n^<(\hat{p}) T^<(\hat{k} - \hat{p}) \frac{d\hat{p}}{(2\pi)^{d+1}} \tag{10}$$

$$B = \frac{ik_n}{k^2 T^<(\hat{k})} \int p^2 v_n^<(\hat{p}) T^<(\hat{k} - \hat{p}) \frac{d\hat{p}}{(2\pi)^{d+1}} \tag{11}$$

$$F_1(\chi_0, \nu_0) = \frac{5\chi_0^4 + 12\chi_0^3\nu_0 - 26\chi_0^2\nu_0^2 - 4\chi_0\nu_0^3 + 13\nu_0^4}{\nu_0^2(\chi_0 - \nu_0)^2(\chi_0 + \nu_0)^3} \tag{12}$$

$$F_2^{(d)}(\chi_0, \nu_0) = \frac{1}{\nu_0^2(\chi_0 - \nu_0)} \left(d^2 - d - 3 + \frac{2\chi_0}{\chi_0 - \nu_0}\right) - \frac{4}{(\chi_0 - \nu_0)(\chi_0 + \nu_0)^2} \left(d^2 - d - 2 + \frac{4\chi_0^2}{\chi_0^2 - \nu_0^2}\right). \tag{13}$$

Taking the limit $r \to 0$, the renormalized diffusivity is obtained from the differential recursion relation:

$$\frac{d\chi(r)}{dr} = \frac{d-1}{d} \frac{S_d}{(2\pi)^d} \frac{\lambda_0'^2 D_0 e^{\varepsilon r}}{\nu(r)[\chi(r) + \nu(r)]\Lambda^\varepsilon} + \frac{2\lambda_0'^3}{d(d+2)} \frac{S_d}{(2\pi)^d} \frac{D_0\chi(r)e^{(\varepsilon+2)r}}{\nu(r)[\chi(r) + \nu(r)]^3\Lambda^{\varepsilon+2}} \times [-2A(r) + B(r)] + \frac{\lambda_0\lambda_0'^2}{2d(d+2)} \frac{S_d}{(2\pi)^d} \frac{e^{(\varepsilon+2)r}}{\Lambda^{\varepsilon+2}} \{-[F_1(\chi, \nu) + F_2^{(d)}(\chi, \nu)]A(r) + F_2^{(d)}(\chi, \nu)B(r)\} \tag{14}$$

where

$$A(r) = \frac{ik_l k_n}{k^2 T^<(\hat{k})} \int_{p<\Lambda e^{-r}} p_l v_n(\hat{p}) T(\hat{k} - \hat{p}) \frac{d\hat{p}}{(2\pi)^{d+1}} \tag{15}$$

$$B(r) = \frac{ik_n}{k^2 T^<(\hat{k})} \int_{p<\Lambda e^{-r}} p^2 v_n(\hat{p}) T(\hat{k} - \hat{p}) \frac{d\hat{p}}{(2\pi)^{d+1}}. \tag{16}$$

Using the RG method again, we can calculate

$$A(r) = 0 \tag{17}$$

$$B(r) = \frac{d-1}{d} \frac{S_d}{(2\pi)^d} \frac{\lambda_0' D_0}{\Lambda^{\varepsilon-2}} \int_r^\infty \frac{e^{(\varepsilon-2)r}}{\nu(r)[\chi(r) + \nu(r)]} dr. \tag{18}$$

When r is large enough, using (3) and (14)–(18) gives (for $y = d = 3$)

$$a = \frac{10}{3} \frac{1}{1+a} + \frac{40}{9} \frac{a}{(1+a)^4} + \frac{20}{9} \frac{2.5a^4 + 6a^3 - 13a^2 - 2a + 6.5}{(a-1)^2(a+1)^4} \tag{19}$$

where

$$a = \lim_{r \to \infty} \frac{\chi(r)}{\nu(r)}.$$

From (19), we calculate $a = 2.125$ so that the renormalized Prandtl number $P_{\text{tr}} = a^{-1} = 0.4706$. Using the calculation for the Kolmogrov constant $C_{\text{K}} = 1.617$ and the relation $\text{Ba} = C_{\text{K}} P_{\text{tr}}$ [1–4], we predict the Batchelor constant to be Ba=0.761 in agreement with the experimental data 0.5–0.8.

In the present work, further evaluations are implemented. Neglecting the fourth term on the right side of (6) yields the prediction for the Batchelor constant to be Ba=1.077, which does not deviate far from the Yakhot–Orszag result Ba=1.161. However, neglecting the third term on the right side of (6), the calculation for the Batchelor constant Ba=0.777 is very close to the result Ba=0.761. Since the random force introduced in the equations (1) obeys the Gaussian distribution, the second and third terms on the right side of (6) only describe the advection of a passive scalar by the turbulent velocity part of the Gaussian distribution. In fact, the turbulent velocity deviates from the Gaussian distribution as a result of the nonlinear mode coupling. The present work shows that the deviation from the Gaussian distribution for the turbulent velocity has important effects upon the advection of a passive scalar.

In the engineering calculation for the turbulent shear flows, the turbulent Prandtl number is approximately 0.9. Kraichnan has calculated the eddy Prandtl number to be $(P_{\text{t}})_{\text{eddy}} = 0.142$ by the direct-interaction approximation [8]. We propose that the renormalized Prandtl number as the eddy Prandtl number is not the same physical quality as the turbulent Prandtl number in the engineering. To understand the relations between them requires further research.

We are grateful to Professor Chao-Hao Gu for numerous helpful suggestions. We would also like to acknowledge Professor Ke-Lin Wang and Professor Bing-Hong Wang for many stimulating discussions of these problems.

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